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L_p Approximation by Analytic Functions*

THOMAS BAGBY

Department of Mathematics, Indiana University, Bloomington, Indiana 47401

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1. INTRODUCTION

In this paper we discuss mean approximation by complex analytic functions in the plane **C**. If $X \subset \mathbf{C}$ is a compact set of positive Lebesgue measure, we denote by $L_p(X)$ the L_p space obtained from Lebesgue measure restricted to X. We will prove the following result.

THEOREM. *If $X \subset \mathbf{C}$ is a compact set of positive Lebesgue measure, $1 < p < \infty$ and $1/p + 1/q = 1$, then the following are equivalent*:

(a) *The rational functions with poles off X are dense in $L_p(X)$.*

(b) *$\gamma_q(G - X) = \gamma_q(G)$ for every bounded open set $G \subset \mathbf{C}$.*

Here the q-capacity γ_q is defined as follows. If $K \subset \mathbf{C}$ is compact, we let

$$\gamma_q(K) = \inf \| u \|_q , \tag{1}$$

where

$$\| u \|_q = \left\{ \iint [| u |^2 + | \operatorname{grad} u |^2]^{q/2} \, dx \, dy \right\}^{1/q} \tag{2}$$

and the infimum is taken over all real-valued functions $u \in C_0^\infty(\mathbf{R}^2)$ such that $u = 1$ on K. If $E \subset \mathbf{C}$ is arbitrary, we define

$$\gamma_q(E) = \sup\{\gamma_q(K) : K \subset E, K \text{ compact}\}.$$

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If $1 \leqslant p < 2$, it is well known that (a) holds if and only if X has no interior; for $p \geqslant 2$ this condition is necessary but not sufficient for (a) to hold. If $p = 2$ the above theorem is a special case of a result of Havin [5], who worked with the fine topology of potential theory. In the present paper we use quasi-topological concepts [4], which are discussed in Section 2. The proof of the theorem is given in Section 3.

Characterizations of (a) in terms of different concepts have been given by Sinanjan [7] and Brennan [1].

Remarks Added in Proof: 1. A sharpened form of Sinanjan's "analytic p-capacity" was used in a study of rational approximation by L. I. Hedberg (Approximation in the Mean by Analytic Functions, Transactions of the American Mathematical Society, to appear). Since the present paper was written, Hedberg has expanded his paper to include a comparison between his capacity and the capacity used here.

2. As the author announced in his talk at the Walsh conference, the quasi-topological methods of the present paper can be extended to obtain a characterization of those compact sets $X \subset \mathbf{C}$ such that the analytic functions on X are dense in $L_p{}^a(X)$, where $L_p{}^a(X)$ denotes the set of all functions in $L_p(X)$ which are analytic on the interior of X. The details will appear in the author's paper, Quasi Topologies and Rational Approximation, in the Journal of Functional Analysis.

2. The Capacity γ_q

If $1 < q < \infty$, we denote by $W_q{}^1$ the Banach space of all functions $u \in L_q(\mathbf{R}^2)$ whose first partial derivatives (in the sense of distribution theory) are also in $L_q(\mathbf{R}^2)$, the norm $\| u \|_q$ being defined by (2). The basic facts about these spaces are given in [6, Chap. 3].

We now discuss some elementary properties of the capacity $\gamma = \gamma_q$ defined above. We begin by noting that the infimum (1) defining the capacity of a compact set K may be taken over all real-valued functions $u \in C_0{}^\infty(\mathbf{R}^2)$ such that $u \geqslant 1$ on K; this can be proved by truncation and use of mollifiers [6]. It is convenient to introduce the outer capacity γ^* of an arbitrary set $E \subset \mathbf{C}$ by

$$\gamma^*(E) = \inf\{\gamma(G) : G \supset E,\ G \text{ open}\}.$$

A set $E \subset \mathbf{C}$ is *capacitable* if $\gamma(E) = \gamma^*(E)$. A property is said to hold *quasi everywhere* (q.e.) if the set where it fails has zero exterior capacity.

From the definitions we see that every open set is capacitable. Moreover, for any decreasing sequence of compact sets $K_j \subset \mathbf{C}$ we have $\gamma(K_j) \to \gamma(\cap K_j)$. It follows that every compact set is capacitable.

From the subadditivity of the norm we get $\gamma(K_1 \cup K_2) \leqslant \gamma(K_1) + \gamma(K_2)$ for compact sets K_1 and K_2. From this follows the countable subadditivity

$$\gamma^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leqslant \sum_{j=1}^{\infty} \gamma^*(E_j)$$

for any sequence of sets $E_j \subset \mathbf{C}$.

The following lemma of Egorov type now follows from an argument of Deny and Lions ([3, Chap. II, Theorem 3.1] or [8, Theorem 4.3]), and we omit the proof.

LEMMA 1. *Suppose that the functions $u_n \in C_0^\infty$ form a convergent sequence in W_q^1. For every $\epsilon > 0$ there exists an open set $G \subset \mathbf{C}$ with $\gamma(G) < \epsilon$ and a subsequence of $\{u_n\}$ converging uniformly off G.*

This lemma motivates the following definitions. A set $E \subset \mathbf{C}$ is *quasi open* if for every $\epsilon > 0$ there exists an open set $G \subset \mathbf{C}$ with $\gamma(G) < \epsilon$ such that $E - G$ is open in $\mathbf{C} - G$. A function $f: \mathbf{C} \to \mathbf{C}$ is *quasi continuous* if for every $\epsilon > 0$ there exists an open set $G \subset \mathbf{C}$ with $\gamma(G) < \epsilon$ such that the restriction of f to $\mathbf{C} - G$ is continuous. Thus if f is quasi continuous and $G \subset \mathbf{C}$ is open, then the inverse image $f^{-1}(G)$ is quasi open. By use of mollifiers, it follows from Lemma 1 that every function in W_q^1 coincides a.e. with a quasi continuous function.

LEMMA 2. *If G is any open set with $\gamma(G) < \infty$, then there exists a real-valued quasi-continuous function $w \in W_q^1$ such that $w \equiv 1$ on G and $\| w \| = \gamma(G)$.*

Remarks. We call w and *equilibrium potential* for G. A related result is given in [8, 4.10].

Proof. Let $K_j = \{x \in G : d(x, \partial G) \geqslant j^{-1}, d(x, 0) \leqslant j\}$. Since $\gamma(K_j) \to \gamma(G)$, we can find real-valued functions $u_j \in C_0^\infty$ such that $u_j = 1$ on K_j and $\| u_j \| \to \gamma(G) < \infty$. From the elementary theory of Sobolev spaces [6] it follows that some subsequence of $\{u_j\}$, which we still call $\{u_j\}$, converges weakly to a limit $u \in W_q^1$. Then $u = 1$ a.e. on G, and by lower semicontinuity of the norm we conclude that $\| u \| \leqslant \gamma(G)$. It is well known that we may approximate u in the W_q^1 norm by a sequence of functions of the form $\psi_j u$, where $\psi_j \in C_0^\infty$ is identically equal to one on $\{| z | < j\}$; applying mollifiers to $\psi_j u$ then gives $\| u \| \geqslant \gamma(G)$. Thus $\| u \| = \gamma(G)$. Finally, applying Lemma 1 to a sequence of such approximants of u gives a quasi-continuous function w of the desired form.

3. Proof of the Theorem

(a) $\Rightarrow$ (b). Suppose that the rational functions with poles off X are dense in $L_p(X)$. Let G be any bounded open set, and let $\phi \in C_0^\infty$ be identically equal to one on $X \cup \bar{G}$. Let w be the equilibrium potential for $G - X$. If $w = 1$ a.e. in G we are done: indeed, approximating w by functions ψw, where $\psi \in C_0^\infty$ is identically equal to one on $\bar{G}$, and applying mollifiers to ψw, we get

$$\gamma(G) \leqslant \| w \| = \gamma(G - X).$$

We therefore assume that $G \cap \{w < 1\}$ has positive Lebesgue measure, and work to obtain a contradiction. Clearly we can find a function $u \in C_0^\infty$ with support in G such that $\{u = 1\} \cap \{w < 1\}$ has positive Lebesgue measure. Now $v = u(\phi - w)$ is a quasi-continuous function in W_q^1 which is identically zero in $\mathbf{C} - X$ but is not the zero function. Since $\pi v = \partial v/\partial \bar{z} * z^{-1}$, we conclude that $\partial v/\partial \bar{z}$ is a nonzero function in $L_q(X)$ which annihilates all rational functions with poles off X, the desired contradiction.

(b) $\Rightarrow$ (a). Suppose $f \in L_q(X)$ annihilates all rational functions with poles off X. Then by the Calderon–Zygmund theory [2] we have $u = \pi^{-1} f * z^{-1} \in W_q^1$ and $f = \partial u/\partial \bar{z}$. Applying Lemma 1 to a sequence of mollifiers of u, we see that u coincides a.e. with a quasi continuous function U such that $U = 0$ on $\mathbf{C} - X$. Now from hypothesis (b) we may easily prove that $\gamma^*(G - X) = \gamma^*(G)$ for every bounded quasi-open set G; in particular, if we take $G = \{U \neq 0\}$ we see that U vanishes q.e. Therefore u is the zero distribution, so $f = \partial u/\partial \bar{z} = 0$.

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